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Smoothing Variational Splines

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Abstract—This paper deals with the problem of constructing some free-form curves and surfaces from given Lagrangian and/or Hermite data. We define the smoothing variational spline function by minimizing a certain quadratic functional in a Sobolev space and establish the convergence of the associated method. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

The theory of the smoothing D^m -splines over an open bounded set is due to Attéia (see, for example, [1]). Likewise, Duchon has chosen the minimization of a quadratic functional as a first approximation of the flexion energy of a thin plate (see [2]). In addition, Arcangéli approximates some Lagrangian or Hermite data by minimizing a functional defined in a Sobolev space (see [3]). Following this line, we generalize these variational methods of approximation to the cases of curves and surfaces, by defining and characterizing the smoothing variational spline function.

Specifically, we present an approximating method for curves and surfaces parameterized by a function, defined on a nonempty bounded open subset, which minimizes a quadratic functional defined in a Sobolev space. Some examples are a fairness functional (see [4,5] for a discrete case), a first approximation of the flexion energy of a thin plate (see [2]), and a functional associated to tangent conditions (see [6] for a discrete case). Moreover, this generic functional consists of different terms associated both to the approximation data and certain restrictions. We notice that the weight assigned to each condition is controlled by a vector of parameters.

In the remainder of this section, we introduce some notations.

First, denote by $\langle \cdot \rangle_{\mathbb{R}^n}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$, respectively, the Euclidean norm and the inner product in \mathbb{R}^n , with $n \in \mathbb{N}^*$, and let $p, m \in \mathbb{N}^*$, and $\mu \in \mathbb{N}$ be such that

$$m > \frac{p}{2} + \mu. \quad (1)$$

Next, let Ω be a bounded open subset of \mathbb{R}^p with Lipschitz-continuous boundary. We denote by $H^m(\Omega; \mathbb{R}^n)$ the usual Sobolev space of (classes of) functions u belonging to $L^2(\Omega; \mathbb{R}^n)$, together

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with all their partial derivatives $\partial^\beta u$ —in the distribution sense—of order $|\beta| \leq m$, where $\beta \in \mathbb{N}^p$. This space is equipped with the norm $\|\cdot\|_{m,\Omega,\mathbb{R}^n}$, the seminorm of order l , $|\cdot|_{l,\Omega,\mathbb{R}^n}$, and the corresponding inner semiproduct of order l , $(\cdot, \cdot)_{l,\Omega,\mathbb{R}^n}$, for $l = 0, \dots, m$.

Finally, we denote by $\mathbb{R}^{N,n}$ the space of real matrices with N rows and n columns, equipped with the inner product $\langle A, B \rangle_{N,n} = \sum_{i,j=1}^{N,n} a_{ij}b_{ij}$, and the corresponding norm $\langle A \rangle = \langle A, A \rangle_{N,n}^{1/2}$.

2. DEFINITION AND CHARACTERIZATION

Let Υ_0 be a curve (if $p = 1$) or a surface (if $p = 2$) parametrized by a function f of $H^m(\Omega; \mathbb{R}^n)$. Suppose we have an ordered finite subset A of distinct points of $\overline{\Omega}$, and an ordered finite set Σ of N_1 linear applications, defined on $H^m(\Omega; \mathbb{R}^n)$, of type $\Phi : v \mapsto \partial^\gamma v(a)$, for $|\gamma| \leq \mu$, with $a \in A$, in such a way, that each point of A is associated with at least one element of Σ . Likewise, we have an other ordered finite set Θ of N_2 continuous inner semiproducts defined on $H^m(\Omega; \mathbb{R}^n)$.

Now, we consider the continuous linear operator L defined from $H^m(\Omega; \mathbb{R}^n)$ into $\mathbb{R}^{N_1,n}$ by $Lv = (\Phi(v))_{\Phi \in \Sigma}$, and the application α defined from $H^m(\Omega; \mathbb{R}^n) \times H^m(\Omega; \mathbb{R}^n)$ into \mathbb{R}^{N_2} by $\alpha(u, v) = (\Psi(u, v))_{\Psi \in \Theta}$. We assume that the operator L satisfies the relation $\ker L \cap P_{m-1}(\Omega; \mathbb{R}^n) = \{0\}$, where $P_{m-1}(\Omega; \mathbb{R}^n)$ is the space of all polynomials with values in \mathbb{R}^n of degree $\leq m-1$.

We consider the following problem: find a parameterization $\sigma \in H^m(\Omega; \mathbb{R}^n)$ of either curve or surface Υ approximating Υ_0 from the data set given by Σ , such that σ minimizes both the seminorms associated to the inner semiproducts of Θ , and the seminorm of order m in $H^m(\Omega; \mathbb{R}^n)$.

EXAMPLE 1. Let Υ_0 be the semicircumference parameterized by the function $f : (0, 1) \rightarrow \mathbb{R}^2$ given by $f(t) = (\cos(\pi t), \sin(\pi t))$. We consider the following data from Υ_0 : $f(0) = (1, 0)$, $f'(0) = (0, \pi)$, $f(0.5) = (0, 1)$, $f(1) = (-1, 0)$, and $f'(1) = (0, -\pi)$, and the inner semiproduct of order 1 defined on $H^2((0, 1); \mathbb{R}^2)$. An example of the above-mentioned problem consists of searching for a function $\sigma \in H^2((0, 1); \mathbb{R}^2)$ such that $\sigma(0)$, $\sigma'(0)$, $\sigma(0.5)$, $\sigma(1)$, $\sigma'(1)$ are close to $f(0)$, $f'(0)$, $f(0.5)$, $f(1)$, $f'(1)$, respectively, and σ minimizes the seminorms $|\cdot|_{1,(0,1),\mathbb{R}^2}$ and $|\cdot|_{2,(0,1),\mathbb{R}^2}$ in $H^2((0, 1); \mathbb{R}^2)$. These seminorms can be interpreted as certain measures of the length and of the curvature, respectively, of a parametric curve.

For all $\tau = (\tau_1, \dots, \tau_{N_2}) \in \mathbb{R}^{N_2}$, with $\tau_i \geq 0$ for any $i = 1, \dots, N_2$, and $\varepsilon > 0$, let $J_{\varepsilon\tau}$ be the functional defined on $H^m(\Omega; \mathbb{R}^n)$ by

$$J_{\varepsilon\tau}(v) = \langle Lv - Lf \rangle_{N_1,n}^2 + \langle \tau, \alpha(v, v) \rangle_{\mathbb{R}^{N_2}} + \varepsilon |v|_{m,\Omega,\mathbb{R}^n}^2.$$

REMARK. We observe that the functional $J_{\varepsilon\tau}(v)$ is composed by three terms:

- the first term indicates how well v approaches the data set $\{\Phi(f), \Phi \in \Sigma\} = \{Lf\}$ in some smoothing least squares sense;
- the second term can represent different conditions, such as fairness conditions (see [4,5]) or tangent conditions (see [6]), while τ weighs the importance given to each of them;
- the third term represents a classical smoothness measure, which is controlled by the parameter ε .

Now we consider the following minimization problem: find $\sigma_{\varepsilon\tau}$ such that

$$\begin{aligned} \sigma_{\varepsilon\tau} &\in H^m(\Omega; \mathbb{R}^n), \\ \forall v \in H^m(\Omega; \mathbb{R}^n), \quad J_{\varepsilon\tau}(\sigma_{\varepsilon\tau}) &\leq J_{\varepsilon\tau}(v). \end{aligned} \quad (2)$$

THEOREM 1. *Problem (2) has a unique solution, called the smoothing variational spline relative to A , L , α , Lf , τ , and ε , which is also the unique solution of the following variational problem: find $\sigma_{\varepsilon\tau}$ such that*

$$\begin{aligned} \sigma_{\varepsilon\tau} &\in H^m(\Omega; \mathbb{R}^n), \\ \forall v \in H^m(\Omega; \mathbb{R}^n), \quad \langle L\sigma_{\varepsilon\tau}, Lv \rangle_{N_1,n} + \langle \tau, \alpha(\sigma_{\varepsilon\tau}, v) \rangle_{\mathbb{R}^{N_2}} + \varepsilon \langle \sigma_{\varepsilon\tau}, v \rangle_{m,\Omega,\mathbb{R}^n} &= \langle Lf, Lv \rangle_{N_1,n}. \end{aligned}$$

PROOF. It suffices to endow $H^m(\Omega; \mathbb{R}^n)$ with the norm

$$[[v]] = \left(\langle Lv \rangle_{N_1, n}^2 + \langle \tau, \alpha(v, v) \rangle_{\mathbb{R}^{N_2}} + |v|_{m, \Omega, \mathbb{R}^n}^2 \right)^{1/2},$$

and then apply the Lax-Milgram Lemma (see [7]).

EXAMPLE 2. We consider the curve, the data set and the inner semiproduct given in Example 1. A solution to that problem can be the smoothing variational spline relative to $A = \{0, 0.5, 1\}$, $L = \{\Phi_i \mid i = 1, \dots, 5\}$, $\alpha = (\cdot, \cdot)_{1, (0,1), \mathbb{R}^2}$, Lf , $\tau \geq 0$ and $\varepsilon > 0$, being $\Phi_1(v) = v(0)$, $\Phi_2(v) = v'(0)$, $\Phi_3(v) = v(0.5)$, $\Phi_4(v) = v(1)$, and $\Phi_5(v) = v'(1)$ for any $v = (v_1, v_2) \in H^2((0, 1); \mathbb{R}^2)$. In other words, the minimum in $H^2((0, 1); \mathbb{R}^2)$ of the following functional:

$$\begin{aligned} J_{\varepsilon\tau}(v) = & (v_1(0) - 1)^2 + v_2(0)^2 + v_1'(0)^2 + (v_2'(0) - \pi)^2 + v_1(0.5)^2 + (v_2(0.5) - 1)^2 \\ & + (v_1(1) + 1)^2 + v_2(1)^2 + v_1'(1)^2 + (v_2'(1) + \pi)^2 \\ & + \tau \int_0^1 (v_1'(t)^2 + v_2'(t)^2) dt + \varepsilon \int_0^1 (v_1''(t)^2 + v_2''(t)^2) dt. \end{aligned}$$

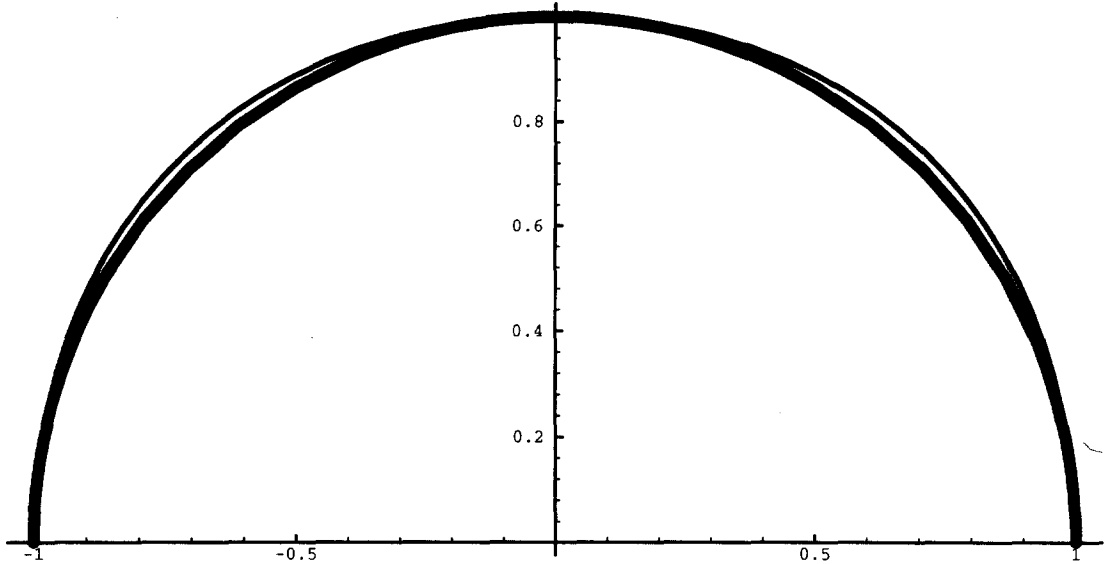


Figure 1. The bold original curve Υ_0 parameterized by f and its approximating curve Υ parameterized by $\sigma_{10^{-5}10^{-5}}$.

The solution of this variational problem is the function $\sigma_{\varepsilon\tau}(t) = (\sigma_1(t), \sigma_2(t))$, where for selected values $\varepsilon = 10^{-5}$, $\tau = 10^{-5}$, and $t \in (0, 0.5)$, we have

$$\begin{aligned} \sigma_1(t) &= 8.83508 - 3.87593 \times 10^{-11} t - 7.83508 \cosh(t) - 7.83507 \times 10^{-5} \sinh(t), \\ \sigma_2(t) &= 1.25272 + 22.6432 t - 1.25249 \cosh(t) - 19.5016 \sinh(t), \end{aligned}$$

while for $t \in (0.5, 1)$, we have

$$\begin{aligned} \sigma_1(t) &= -8.83508 - 3.88692 \times 10^{-11} t + 12.0902 \cosh(t) - 9.20791 \sinh(t), \\ \sigma_2(t) &= 23.8959 - 22.6432 t - 24.851 \cosh(t) + 31.5645 \sinh(t). \end{aligned}$$

The smoothing variational spline obtained in Example 2 is a classical spline under tension (see [8]).

REMARK. For $Lv = (v(a))_{a \in A}$, with $v \in H^m(\Omega)$ and $\tau = 0$, $\sigma_{\varepsilon 0}$ is a smoothing D^m -spline in a classical sense (see [3]).

3. CONVERGENCE

We shall show that, under adequate hypotheses, the smoothing variational spline function converges to f . Let \mathcal{D} be a set of real positive numbers, where 0 is an accumulation point, and suppose that for any $d \in \mathcal{D}$, we have Σ , Θ , A , L , α , ε , τ , and $J_{\varepsilon\tau}$ (depending on d). For any $d \in \mathcal{D}$, let $\sigma_{\varepsilon\tau}^d$ be the smoothing variational spline relative to A , L , α , Lf , τ , and ε which minimizes the functional $J_{\varepsilon\tau}$ in $H^m(\Omega; \mathbb{R}^n)$ and suppose that

$$\sup_{x \in \Omega} \max_{a \in A_{LG}} \langle x - a \rangle_{\mathbb{R}^p} = d, \quad (3)$$

where A_{LG} is a subset of A such that for all $a \in A_{LG}$, there exists $\Phi \in \Sigma$ satisfying $\Phi(v) = v(a)$, for all $v \in H^m(\Omega; \mathbb{R}^n)$, i.e., the points of A associated to the Lagrangian data.

THEOREM 2. *Suppose that hypotheses (1) and (3) hold, and that*

$$\langle \tau \rangle_{\mathbb{R}^{N_2}} = o(\varepsilon), \quad d \rightarrow 0 \quad (4)$$

and

$$\varepsilon = o(d^{-p}), \quad d \rightarrow 0. \quad (5)$$

Then

$$\lim_{d \rightarrow 0} \|\sigma_{\varepsilon\tau}^d - f\|_{m, \Omega, \mathbb{R}^n} = 0.$$

PROOF. First, for any $d \in \mathcal{D}$, we have $J_{\varepsilon\tau}(\sigma_{\varepsilon\tau}^d) \leq J_{\varepsilon\tau}(f)$. Thus,

$$|\sigma_{\varepsilon\tau}^d|_{m, \Omega, \mathbb{R}^n}^2 \leq \frac{\langle \tau \rangle_{\mathbb{R}^{N_2}}}{\varepsilon} \langle \alpha(f, f) \rangle_{\mathbb{R}^{N_2}} + |f|_{m, \Omega, \mathbb{R}^n}^2,$$

and by applying (4), we obtain

$$|\sigma_{\varepsilon\tau}^d|_{m, \Omega, \mathbb{R}^n}^2 = o(1) + |f|_{m, \Omega, \mathbb{R}^n}^2, \quad d \rightarrow 0. \quad (6)$$

Next, for all $d \in \mathcal{D}$, we have

$$\langle L(\sigma_{\varepsilon\tau}^d - f) \rangle_{N_1, n}^2 \leq \langle \tau, \alpha(f, f) \rangle_{\mathbb{R}^{N_2}} + \varepsilon |f|_{m, \Omega, \mathbb{R}^n}^2,$$

and we can deduce that there exists $C > 0$ such that

$$\langle L(\sigma_{\varepsilon\tau}^d - f) \rangle_{N_1, n}^2 \leq C \langle \tau \rangle_{\mathbb{R}^{N_2}} \|f\|_{m, \Omega, \mathbb{R}^n}^2 + \varepsilon |f|_{m, \Omega, \mathbb{R}^n}^2.$$

Then, for any $d \in \mathcal{D}$, it follows from (4) that

$$\langle L(\sigma_{\varepsilon\tau}^d - f) \rangle_{N_1, n}^2 = \varepsilon \left(o(1) + |f|_{m, \Omega, \mathbb{R}^n}^2 \right). \quad (7)$$

By taking into account (5), (6), and (7) and reasoning as in the proof of Theorem 6.3 in [9], we deduce that

$$\exists C > 0, \quad \exists \eta > 0, \quad \forall d \in \mathcal{D}, \quad d \leq \eta, \quad \|\sigma_{\varepsilon\tau}^d\|_{m, \Omega, \mathbb{R}^n} \leq C.$$

Hence, the family $(\sigma_{\varepsilon\tau}^d)_{\substack{d \in \mathcal{D} \\ d \leq \eta}}$ is bounded in $H^m(\Omega; \mathbb{R}^n)$. Therefore, there exists a sequence $(\sigma_{\varepsilon_l \tau_l}^{d_l})_{l \in \mathbb{N}}$ extracted from this family, with $\lim_{l \rightarrow +\infty} d_l = 0$, $\varepsilon = \varepsilon(d_l)$, $\tau = \tau(d_l)$, and an element f^* of $H^m(\Omega; \mathbb{R}^n)$ such that $(\sigma_{\varepsilon_l \tau_l}^{d_l})_{l \in \mathbb{N}}$ converges weakly to f^* in $H^m(\Omega; \mathbb{R}^n)$, when $l \rightarrow +\infty$. Finally, the result is obtained by following the procedure used in items 3–5 of Theorem 6.3 in [9].

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